

# ON NESTED SEQUENCES OF CONVEX SETS IN A BANACH SPACE

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**ABSTRACT.** In this paper we study different aspects of the representation of weak\*-compact convex sets of the bidual  $X^{**}$  of a separable Banach space  $X$  via a nested sequence of closed convex bounded sets of  $X$ .

## 1. INTRODUCTION

In this paper we solve several problems about nested intersections of convex closed bounded sets in Banach spaces.

We begin with a study of different aspects of the representation of weak\*-compact convex sets of the bidual  $X^{**}$  of a separable Banach space  $X$  via a nested sequence of closed convex bounded sets of  $X$ . Precisely, let us say that a convex closed bounded subset  $C \subset X^{**}$  is *representable* if it can be written as the intersection

$$C = \bigcap_{n \in \mathbb{N}} \overline{C_n}^{w*}$$

of a nested sequence  $(C_n)$  of bounded convex closed subsets of  $X$ . This topic was considered in [8, 9], where the problem of which weak\*-closed convex sets of the bidual are representable was posed. In [6], Bernardes shows that when  $X^*$  is separable every weak\*-compact convex subset of  $X^{**}$  is representable. Here we will show that compact convex sets of  $X^{**}$  are representable if and only if  $X$  does not contain  $\ell_1$  and also that there are spaces without copies of  $\ell_1$  containing weak\*-compact convex metrizable subsets of the bidual not representable.

In Section 3, we solve problem (2) in [9] showing that when the sets are viewed as the distance type (in the sense of [8]) they define, i.e., as elements of  $\mathbb{R}^X$ , then every weak\*-compact convex set  $C \subset X^{**}$  is represented by a nested sequence  $(C_n)$  of closed convex sets of  $X$ ; which means that for all  $x \in X$

$$\text{dist}(x, C) = \lim \text{dist}(x, C_n).$$

In Section 4 we present two examples: the first one solves Marino's question [17] about the possibility of enlarging nested sequences of convex sets to get better

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intersections; the second one solves Behrends' question about the validity of  $\varepsilon = 0$  in the Helly-Bárány theorem [4].

## 2. REPRESENTATION OF CONVEX SETS IN BIDUALS

*Definition 1.* A Banach space is said to enjoy the Convex Representation Property, in short CRP (resp. Compact Convex Representation Property, in short CCRP) if every weak\*-compact (resp. compact) convex subset  $C$  of  $X^{**}$  can be represented as the intersection

$$C = \bigcap_{n \in \mathbb{N}} \overline{C_n}^{w*}$$

of a nested sequence  $(C_n)$  of bounded convex closed subsets of  $X$ .

**Proposition 1.** *A separable Banach space has CCRP if and only if it does not contain  $\ell_1$ .*

*Proof.* The necessity follows from the Odell-Rosenthal characterization [19] of separable Banach spaces containing  $\ell_1$ . Indeed, if  $X$  contains  $\ell_1$  then there is an element  $\mu \in X^{**}$  which is not the weak\*-limit of any sequence of elements of  $X$ . Hence,  $\{\mu\} = \bigcap_{n \in \mathbb{N}} \overline{C_n}^{w*}$  is impossible: taking an element  $c_n \in C_n$  one would get  $\emptyset \neq \bigcap_n \overline{\{c_k : k \geq n\}}^{w*} \subset \bigcap_n \overline{C_n}^{w*} = \{\mu\}$ , which means that  $\mu$  is the only weak\*-cluster point of the sequence  $(c_n)$  and thus  $\mu = w^* - \lim c_n$ .

As for the sufficiency, let  $K$  be a compact convex subset of  $X^{**}$ . For every  $n \in \mathbb{N}$ , let  $F_n = \{z_n^k : k \in I_n\}$  be a finite subset of  $K$  for which  $K \subset F_n + n^{-1}B_{X^{**}}$ . There is no loss of generality assuming that  $F_n \subset F_{n+1}$ . For each  $z_n^k \in F_n$ , let  $(x_n^k(m))_m \subset X$  be a sequence in  $X$  weak\*-convergent to  $z_n^k$ . Set

$$C_n = \overline{\text{conv}}\{x_n^k(m), k \in I_n, m \geq n\} + n^{-1}B_X$$

It is clear that  $C_n$  is a nested sequence of closed convex sets of  $X$ . Moreover,

$$K \subset F_n + n^{-1}B_{X^{**}} \subset \overline{\text{conv}}\{x_n^k(m), k \in I_n, m \geq n\} + n^{-1}B_X^{w*} = \overline{C_n}^{w*},$$

and thus  $K \subset \bigcap_n \overline{C_n}^{w*}$ .

Fix now  $p \in \bigcap_n \overline{C_n}^{w*}$ . Since  $p \in \overline{C_n}^{w*}$ , there is a finite convex combination  $\sum_{i \in I_n} \theta_i z_n^i$  for which  $\|p - \sum_{i \in I_n} \theta_i z_n^i\| \leq n^{-1}$ . This implies that  $p \in \overline{K} = K$  and thus  $\bigcap_n \overline{C_n}^{w*} \subset K$ .  $\square$

This shows that Problem 1 in [9] has a negative answer. On the other hand, Bernardes obtains in [6] an affirmative answer when  $X^*$  is separable, which is somehow the best that can be expected. Let us briefly review and extend Bernardes' result. Recall that a partially ordered set  $\Gamma$  is called filtering when for any two points  $i, j \in \Gamma$  there is  $k \in \Gamma$  such that  $i \leq k$  and  $j \leq k$ . An indexed family of

subsets  $(C_\alpha)_{\alpha \in \Gamma}$  will be called filtering when it is filtering with respect to the natural (reverse) order; i.e., whenever  $\alpha \leq \beta$  then  $C_\beta \subset C_\alpha$ . One has:

**Proposition 1.** *If  $C$  is a convex weak\*-compact set in the bidual  $X^{**}$  of a Banach space  $X$  then there is a filtering family  $(C_\alpha)_{\alpha \in \Gamma}$  of convex bounded and closed subsets of  $X$  such that*

$$C = \bigcap_{\alpha \in \Gamma} \overline{C_\alpha}^{w*}.$$

*Proof.* There is no loss of generality assuming that  $C \subset B_{X^{**}}$ . Let  $\Gamma$  be the partially ordered set of finite subsets of  $B_{X^*}$ . For each  $\alpha \in \Gamma$  we denote  $|\alpha|$  the cardinal of the set  $\alpha$ . Set now

$$C_\alpha = \{x \in X : \exists z \in C : \forall y \in \alpha : |(z - x)(y)| \leq |\alpha|^{-1}\}.$$

This family  $(C_\alpha)_{\alpha \in \Gamma}$  is filtering, as well as  $(\overline{C_\alpha}^{w*})_{\alpha \in \Gamma}$ , which ensures that  $\bigcap_{\alpha \in \Gamma} \overline{C_\alpha}^{w*}$  is nonempty. Let us show the equality

$$C = \bigcap_{\alpha \in \Gamma} \overline{C_\alpha}^{w*}.$$

- $C \subset \bigcap_{\alpha \in \Gamma} \overline{C_\alpha}^{w*}$ : Let  $z \in C \subset B_{X^{**}}$ ; given  $\alpha \in \Gamma$ , by the Banach-Alaoglu theorem, there is  $x \in B_X$  such that  $|(z - x)(y)| < |\alpha|^{-1}$  for all  $y \in \alpha$ . Hence  $x \in C_\alpha$  and thus  $z \in \overline{C_\alpha}^{w*}$ .
- $\bigcap_{\alpha \in \Gamma} \overline{C_\alpha}^{w*} \subset C$ : Let  $z \in \bigcap_{\alpha \in \Gamma} \overline{C_\alpha}^{w*}$  and let  $V_{\alpha, \varepsilon}$  be the weak\*-neighborhood of 0 determined by  $\alpha \in \Gamma$  and  $\varepsilon > 0$ ; i.e.,  $V_{\alpha, \varepsilon} = \{p \in X^{**} : \forall y \in \alpha : |p(y)| \leq \varepsilon\}$ . Pick  $\beta \in \Gamma$  with  $\alpha \leq \beta$  and  $|\beta|^{-1} \leq \varepsilon$ . Since  $z \in \overline{C_\beta}^{w*}$ , there is  $x \in C_\beta$  such that  $|(z - x)(y)| \leq \varepsilon$  for all  $y \in \alpha$ ; which moreover means that there is some  $z' \in C$  such that  $|(z' - x)(y)| \leq |\beta|^{-1} \leq \varepsilon$  for all  $y \in \beta$ . Putting all together one gets that for  $y \in \alpha$

$$|(z - z')(y)| = |(z - x)(y) + (x - z')(y)| \leq 2\varepsilon$$

and thus  $z - z' \in V_{\alpha, 2\varepsilon}$ . Hence  $z \in \overline{C}^{w*} = C$ .

□

The size of  $\Gamma$  can be reduced just taking first a dense subset  $Y \subset B_{X^*}$  and then fixing as  $\Gamma$  a fundamental family of finite sets of  $Y$ , in the sense that every finite subset of  $Y$  is contained in some element of  $\Gamma$ . Such reduction modifies the proof as follows: from the first finite set  $\alpha$ —no longer in  $\Gamma$ —determining  $V_{\alpha, \varepsilon}$  one must take a set  $\beta \in \Gamma$  such that for each  $y \in \alpha$  there is  $y' \in \beta$  so that  $\|y - y'\| \leq |\beta|^{-1} \leq \varepsilon$ . Get  $x$  and  $z'$  as above. Finally, for  $y \in \alpha$ , one gets

$$|(z - z')(y)| = |(z - z')(y - y') + (z - z')(y')| \leq \varepsilon + 2\varepsilon = 3\varepsilon.$$

The consequence of such simplification is that when  $X^*$  is separable then  $\Gamma$  reduces to  $\mathbb{N}$  and thus one gets the main result in [6]:

**Corollary 1** (Bernardes). *Every Banach space with separable dual has CRP.*

One therefore has:

$$X^* \text{ separable} \implies CRP \implies CCRP \iff \ell_1 \not\subseteq X.$$

This suggests two questions: 1) whether CCRP implies CRP and 2) whether CRP implies having separable dual. One has

**Proposition 2.** *CCRP does not imply CRP.*

To prove this we are going to show that the James-Tree space –perhaps the simplest space not containing  $\ell_1$  but having nonseparable dual– fails CRP. For information about  $JT$ , we refer to [13, Chapter VIII]. We begin with a preparatory lemma that can be considered as a complement to Kalton [15, Lemma 5.1].

**Lemma 1.** *Let  $(C_n)_n$  be a nested sequence of bounded closed convex subsets of a Banach space  $X$ . If  $\bigcap_n \overline{C_n}^{w^*}$  is weak\*-metrizable then:*

- (1) *Every  $g \in \bigcap_n \overline{C_n}^{w^*}$  is the weak\*-limit of a sequence  $(c_n)$  with  $c_n \in C_n$ .*
- (2) *Every sequence  $(c_n)$  with  $c_n \in C_n$  admits a weak\*-convergent subsequence.*

*Proof.* (1) is clear: let  $(V_n)_n$  be a sequence of weak\*-neighborhoods of  $g$  such that  $\{g\} = \bigcap V_n \cap \bigcap_n \overline{C_n}^{w^*}$ . Picking  $c_n \in C_n \cap V_n$  one gets  $\{g\} = \overline{\{c_n\}}^{w^*}$ .

To prove (2), let us consider the equivalence relation on the set  $\mathcal{P}_\infty(\mathbb{N})$  of infinite subsets of  $\mathbb{N}$ :  $A \sim B$  if and only if  $A$  and  $B$  coincide except for a finite set. Moreover,  $K$  will denote the set of all compact subsets of  $\bigcap_n \overline{C_n}^{w^*}$ . Given a sequence  $(c_n)$  with  $c_n \in C_n$  we define the following map  $w : \mathcal{P}_\infty(\mathbb{N}) / \sim \rightarrow K$ :

$$w([A]) = \overline{\bigcap_k \{c_n : n \in A, n > k\}}^{w^*}.$$

The set  $\mathcal{P}_\infty(\mathbb{N}) / \sim$  admits a natural order as  $[A] \leq [B]$  if  $A$  is eventually contained in  $B$ . This order has the property that for every decreasing sequence  $([A_n])_n$  there is an element  $[B]$  with  $[B] \leq [A_n]$  for all  $n$ . Since  $\bigcap_n \overline{C_n}^{w^*}$  is metrizable, it follows [3, Sect. 2] that there is  $M \in \mathcal{P}_\infty(\mathbb{N})$  on which  $w$  is stationary; i.e.,  $w([C]) = w([M])$  for all infinite subsets  $C \subset M$ . This immediately yields that  $w(\{c_n\}_{n \in M})$  has only one point, and thus  $\{c_n\}_{n \in M}$  is weak\*-convergent.  $\square$

Let us denote by  $G$  the set of all branches of the dyadic tree  $T$ . For each  $r \in G$ , let  $e_r$  denote the corresponding element of the basis of  $\ell_2(G)$  considered as a subspace of  $JT^{**}$ . Let  $\{e_{k,l} : k \in \mathbb{N}_0, 1 \leq l \leq 2^k\}$  denote the unit vector basis of  $JT$ . The action of  $e_r$  on  $x^* \in JT^*$  is given by

$$\langle x^*, e_r \rangle = \lim_{\text{along } r} \langle e_{k,l}, x^* \rangle.$$

For each  $m \in \mathbb{N}$  we denote by  $P_m$  the norm-one projection in  $JT$  defined by  $P_m e_{k,l} = e_{k,l}$  if  $k \geq m$ , and  $P_m e_{k,l} = 0$  otherwise. For each  $r \in G$  we consider  $f_r \in JT^*$  given by  $\langle e_{k,l}, f_r \rangle$  equal to 1 if  $(k,l) \in r$ , and equal to 0 otherwise. Observe that  $\langle f_r, e_s \rangle = \delta_{r,s}$ . Let  $S = \{s_n : n \in \mathbb{N}\}$  denote a countable subset of  $G$  such that the branches in  $S$  include all the nodes of the tree  $T$ .

**Proof of Proposition 2.** Let us show that the closed unit ball  $B$  of  $\ell_2(S)$  cannot be represented. Assume that we can write  $B = \bigcap_{n \in \mathbb{N}} \overline{C_n}^{w^*}$ . The set  $B$  is  $w^*$ -metrizable, because it is the unit ball of a separable reflexive subspace. By Lemma 1, each vector in  $B$  is the  $w^*$ -limit of a sequence  $(x_n)$  with  $x_n \in C_n$ . For each  $s \in S$  we select  $x_n^s \in C_n$  so that  $w^*\text{-}\lim x_n^s = e_s$ . Note that  $\lim_n \|(I - P_k)x_n^s\| = 0$  for every  $k$  and  $s$ .

We take  $t_1 \in S$ ,  $t_1 \neq s_1$ . Also we take  $x_1 = x_{n_1}^{t_1}$  with  $|\langle x_1, f_{t_1} \rangle - 1| < 2^{-1}$ , and select  $(k_1, l_1) \in t_1 \setminus s_1$  such that  $\|P_{k_1} x_1\| < 2^{-1}$ . Next we take  $t_2 \in S$  with  $(k_1, l_1) \in t_2$  and  $t_2 \neq s_2$ . Also we take  $x_2 = x_{n_2}^{t_2}$  with  $\|(I - P_{k_1})x_2\| < 2^{-2}$  and  $|\langle x_2, f_{t_2} \rangle - 1| < 2^{-2}$ , and select  $(k_2, l_2) \in t_2 \setminus s_2$  with  $k_2 > k_1$  such that  $\|P_{k_2} x_2\| < 2^{-2}$ . Proceeding in this way we obtain a sequence  $(x_i)$  that is eventually contained in each  $C_n$  and an ordered sequence of different nodes  $(k_i, l_i)$  that determine a branch  $r \in G \setminus S$ . Since  $JT$  is separable and contains no copies of  $\ell_1$ , the sequence  $(x_i)$  has a subsequence that is  $w^*$ -convergent to some  $x^{**} \in JT^{**}$  [12, First Theorem in p. 215]. Thus,  $x^{**} \in \bigcap_{n \in \mathbb{N}} \overline{C_n}^{w^*}$ , but  $x^{**} \notin B$  since  $\langle f_r, x^{**} \rangle = 1$ .  $\square$

Proposition 1 thus characterizes the CCRP, while Proposition 2 shows that even when compact convex sets are representable, arbitrary weak\*-metrizable convex bounded closed sets do not have to be. The question of which convex sets are representable thus arises. Bigger than compact spaces are the so called small sets [5, 10, 1], but it was shown in [5] that a closed bounded convex small set is compact.

### 3. REPRESENTATION OF CONVEX SETS IN THE HYPERSPACE

The theory of types in Banach spaces represents the elements of a Banach space  $g \in X$  as functions  $\tau_g(x) = \|x - g\|$ . These are the elementary types and the types are the closure of the set of elementary types in  $\mathbb{R}^X$ . It can be shown that bidual types, i.e., functions having the form  $\tau_g(x) = \|x - g\|$  for  $g \in X^{**}$  are also types [14]. In close parallelism, the theory of distance types was developed in [8]: in it, the elements to be represented are the closed bounded convex subsets  $C$  of  $X$  via the function  $d_C(x) = \text{dist}(x, C)$ . These are the elementary distance types. The  $\emptyset$ -distance types are the functions of the form  $d(x) = \lim d_{C_n}(x)$  where  $(C_n)$  is a nested sequence of closed bounded convex subsets of  $X$  with empty intersection. In [8, Thm. 4.1] it was shown the existence of  $\emptyset$ -distance types that are not types in every nonreflexive separable Banach space. It was also shown [8, Thm. 5.1] that

bidual types on separable Banach spaces coincide with  $\emptyset$ -distance types defined by "flat" (in the sense of Milman and Milman [18]) nested sequences of bounded convex closed sets  $(C_n)$ . In [9, Thm. 1] it is shown that given a nested sequence  $(C_n)$  of bounded convex closed sets on a separable space  $X$  one always has

$$\text{dist}(x, \bigcap \overline{C_n}^{w*}) = \lim \text{dist}(x, C_n).$$

While Bernardes shows in [6, Thm. 1] that that happens in all Banach spaces.

All this suggests the problem [9, Problem 2] whether the analogue of Farmaki's (bidual types are types) also holds for distance types; i.e., if given a weak\*-compact convex subset  $C$  of  $X^{**}$ , the *bidual distance type* it defines  $d_C(x) = \text{dist}(x, C)$  on  $X$ , is a  $\emptyset$ -distance type. Let us give an affirmative answer.

**Proposition 3.** *Let  $C$  be a weak\*-compact convex subset of the bidual  $X^{**}$  of a separable space  $X$  such that  $C \cap X = \emptyset$ . There is a nested sequence  $(C_n)$  of closed convex sets of  $X$  such that  $C \subset \bigcap_n \overline{C_n}^{w*}$  and for all  $x \in X$*

$$\text{dist}(x, C) = \lim \text{dist}(x, C_n).$$

*Proof.* Let  $(x_n)$  be a dense sequence in  $X$ . Since  $C$  is bounded, it is contained in the ball  $\gamma B_{X^{**}}$  for some  $\gamma > 0$ . We proceed inductively: pick  $x_1$ , let  $\alpha_1 = \text{dist}(x_1, C)$ , then set a monotone increasing sequence  $(\alpha_n^1)$  convergent to  $\alpha_1$ . Pick functionals  $\varphi_n^1 \in B_{X^*}$  that strictly separate  $C$  and  $x_1 + (\alpha_n^1)B_{X^{**}}$ , say

$$\inf_{z \in C} z(\varphi_n^1) > \|x_1\| + \alpha_n^1 + 2\varepsilon_n^1.$$

Set  $C_{n,1} = \{x \in X : \exists z \in C : |(z - x)(\varphi_n^1)| \leq \varepsilon_n^1\} \cap \gamma B_{X^{**}}$ . The sequence of convex sets  $C_{n,1}$  is nested and every point  $z \in C$  belongs to the weak\*-closure of some set  $\{x \in X : |(z - x)(\varphi_n^1)| \leq n^{-1}\}$  which is in turn contained in  $C_{n,1}$ . Thus,  $C \subset \bigcap_n \overline{C_{n,1}}^{w*}$ . Moreover,  $x_1 + (\alpha_1)B_{X^{**}} \cap \bigcap_n \overline{C_{n,1}}^{w*} = \emptyset$  because otherwise there should be elements  $c_n \in C_{n,1}$  for which  $(x_1 + \alpha_1 b - c_n)(\varphi_n^1) < \varepsilon_n^1$ ; since there must be  $z_n \in C$  for which  $|(z_n - c_n)(\varphi_n^1)| \leq \varepsilon_n^1$ , pick  $z \in C$  a weak\*-accumulation point of  $(z_n)$  to conclude that  $(x_1 + \alpha_1 b - z)(\varphi_n^1) = (x_1 + \alpha_1 b - c_n + c_n - z)(\varphi_n^1) \leq 2\varepsilon_n^1$  which immediately yields

$$z(\varphi_n^1) = (x_1 + \alpha_1 b)(\varphi_n^1) - (x_1 + \alpha_1 b - z)(\varphi_n^1) \leq \|x_1\| + \alpha_n^1 + 2\varepsilon_n^1$$

in contradiction with the separation above.

Thus, by [9, Thm. 1] we get  $\text{dist}(x_1, C) = \text{dist}(x_1, \bigcap \overline{C_{n,1}}^{w*}) = \lim \text{dist}(x_1, C_{n,1})$ .

We pass to  $x_2$ . Everything goes as before except that all the action is going to happen inside  $\bigcap \overline{C_{n,1}}^{w*}$ . Precisely, once  $\alpha_2, \alpha_n^2, \varphi_n^2, \varepsilon_n^2$  have been fixed by the same procedure as above, set

$$C_{n,2} = \{x \in X : \exists z \in C : \max_{i=1,2} |(z - x)(\varphi_n^i)| \leq \varepsilon_n^i\} \cap \gamma B_{X^{**}}$$

to conclude that  $C \subset \bigcap_n \overline{C_{n,2}}^{w*} \subset \bigcap_n \overline{C_{n,1}}^{w*}$  and  $\text{dist}(x_i, C) = \text{dist}(x_i, \bigcap \overline{C_{n,2}}^{w*}) = \lim \text{dist}(x_i, C_{n,2})$  for  $i = 1, 2$ . Proceed inductively. Since  $C_{n,k+1} \subset C_{n,k}$ , we can diagonalize the final sequence of sequences to get the sequence  $(C_{k,k})$ , which satisfies  $C \subset \bigcap \overline{C_{k,k}}^{w*}$  and, moreover, for all  $n$  one has

$$\text{dist}(x_n, C) = \text{dist}(x_n, \bigcap \overline{C_{k,k}}^{w*}) = \lim \text{dist}(x_n, C_{k,k}).$$

By continuity, the equality remains valid for all  $x \in X$ .  $\square$

In the classical case, as Farmaki remarks in [14], it is not obvious that *fourth-dual* types, i.e., applications having the form  $\tau_g(x) = \|x + g\|$  for  $g \in X^4$  on separable spaces  $X$  are necessarily types. One thus may ask: Let  $X$  be a separable Banach space and let  $C \subset X^{2k}$  be a bounded weak\*-closed convex. Must there be a sequence  $(C_n)$  of bounded convex closed subset of  $X$  such that for every  $x \in X$  one has  $\text{dist}(x, C) = \lim \text{dist}(x, C_n)$ ?

#### 4. FURTHER PROPERTIES OF NESTED SEQUENCES

**4.1. Enlarging sets for better intersection: Marino's problem.** Let  $A$  be a closed set. For  $\varepsilon > 0$  we set

$$A^\varepsilon = \{x \in X : \text{dist}(x, A) \leq \varepsilon\}.$$

An extremely nice result of Marino [17] establishes that given any family  $(G_\gamma)$  of convex sets with nonempty intersection then either  $\bigcap_\gamma G_\gamma^\varepsilon$  is bounded for every  $\varepsilon > 0$  or is unbounded for every  $\varepsilon > 0$ . A question left open in [9, p.583] is whether it is possible to have  $\bigcap A_n = \emptyset$ , some intersections  $\bigcap A_n^\varepsilon$  nonempty and bounded and others unbounded. The next example shows it can be so:

**Example 2.** Consider in  $\ell_1$  the sequence  $A_{2k} = \{x \in \ell_1 : x_{k+1} \leq -\frac{2^{k+1}-1}{2^{k+1}}\}$  and  $A_{2k-1} = \{x \in \ell_1 : x_k \geq 1\}$ . Then  $\bigcap A_n = \emptyset = \bigcap A_n^\varepsilon = \emptyset$  for all  $\varepsilon < 1$ , while

$$\bigcap A_n^1 = \{x \in \ell_1 : \forall k \ 0 \leq x_k \leq \frac{1}{2^k}\}$$

and  $\bigcap A_n^{1+\varepsilon}$  is unbounded for all  $\varepsilon > 0$  since all  $x \in \ell_1$  with  $-\varepsilon \leq x_i \leq 0$  for every  $i$  belong to that set.

The choice of  $\ell_1$  for the example is not at random: during the proof of [9, Prop. 9] it is shown that in reflexive spaces,  $\bigcap A_n = \emptyset$  implies  $\bigcap A_n^\varepsilon = \emptyset$  for all  $\varepsilon > 0$ . Marino's theorem in combination with [9, Prop. 9] yields that in a non-reflexive space if  $\alpha = \inf\{\varepsilon > 0 : \bigcap A_n^\varepsilon \neq \emptyset\}$  then either  $\bigcap A_n^\varepsilon$  is bounded for all  $\varepsilon > \alpha$  or unbounded for all  $\varepsilon > \alpha$ . Let us show now that Marino's theorem remains "almost" valid for nested sequences with empty intersection in a finite dimensional space. In this case, the boundedness of some  $A_n$  immediately implies, by compactness, that

$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$ . Assume thus one has a nested sequence of unbounded convex sets. Let  $T_k = \{x \in X : k \leq \|x\| \leq k+1\}$ . One has

**Lemma 2.** *Let  $(A_n)$  be a sequence of unbounded connected sets in a finite dimensional space  $X$ . Then either  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$  or for all but finitely many  $k \in \mathbb{N}$  and every  $\varepsilon > 0$  there is an infinite subset  $N_k \subset \mathbb{N}$  such that  $T_k \cap \bigcap_{n \in N_k} A_n^\varepsilon \neq \emptyset$ .*

*Proof.* If for every  $k \in \mathbb{N}$  the ball  $kB$  of radius  $k$  does not intersect  $\bigcap_{n \in \mathbb{N}} A_n$  then  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ . Otherwise, let  $x_{n,k} \in A_n \cap kB$ . Since  $A_n$  is unbounded, there is a point  $y_{n,k+1}$  with  $\|y_{n,k+1}\| > k+1$ . Since  $A_n$  is connected, there is some  $x_{n,k+1} \in A_n$  with  $k \leq \|x_{n,k+1}\| \leq k+1$ , and thus in  $A_n \cap T_k$ . The sequence  $(x_{n,k+1})_n$  lies in the compact set  $T_k$  and thus for some infinite subset  $N_k \subset \mathbb{N}$  the subsequence  $(x_{n,k+1})_{n \in N_k}$  is convergent to some point  $x_{k+1} \in T_k$ . Thus,  $x_{k+1} + \varepsilon B$  intersects the sets  $\{A_n : n \in N_k\}$  and thus  $\bigcap_{n \in N_k} A_n^\varepsilon \cap T_k \neq \emptyset$ .  $\square$

Thus we get:

**Proposition 2.** *Let  $(A_n)$  be a nested sequence of unbounded connected sets in a finite dimensional space  $X$ . Then either  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$  or  $\bigcap_{n \in \mathbb{N}} A_n^\varepsilon$  is unbounded for every  $\varepsilon > 0$ .*

The assertion obviously fails for non-connected sets and also fails in infinite dimensional spaces:

**Example 1.** In  $\ell_2$  take  $A_n = \{x \in \ell_2 : \forall k > n, 0 \leq x_k \leq 1 \text{ and } \forall k \leq n, x_k = 0\}$ . This is a nested sequence of unbounded convex closed sets such that  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$  while for all  $\varepsilon > 0$  the set  $\bigcap_{n \in \mathbb{N}} A_n^\varepsilon$  is bounded: indeed, if  $y \in A_n^\varepsilon$  for all  $n$  then there is  $x_n \in A_n$  for which  $\|y - x_n\| \leq \varepsilon$ ; thus  $\sum_{i=1}^n |y_i|^2 \leq \varepsilon^2$  for all  $n$ , so  $\|y\| \leq \varepsilon^2$ .

**4.2. On the Helly-Bárány theorem.** In one of the main theorems in [4] Behrends establishes a Helly-Bárány theorem for separable Banach spaces [4, Thm. 5.5]: *Let  $X$  be a separable Banach space and  $\mathcal{C}_n$  a family of nonvoid, closed and convex subsets of the unit ball  $B$  for every  $n$ . Suppose that there is a positive  $\varepsilon_0 \leq 1$  such that  $\bigcap_{C \in \mathcal{C}_n} C + \varepsilon B = \emptyset$  for every  $n$  and every  $0 < \varepsilon < \varepsilon_0$ . Then there are  $C_n \in \mathcal{C}_n$  such that  $\bigcap_n C_n + \varepsilon B = \emptyset$ .* Behrends asks [4, Remark 2, p. 17] whether one can put  $\varepsilon = \varepsilon_0$  in the previous theorem. The following example shows that the answer is no:

**Example.** In  $c_0$ , the family  $\mathcal{C}_n$  contains two convex sets:

$$a_n^+ = \{x \in c_0 : \forall i \in \mathbb{N} : |x_i| \leq \frac{1}{2}(1 + \frac{1}{i}) \text{ and } |x_n| = \frac{1}{2}(1 + \frac{1}{n})\}$$

and

$$a_n^- = \{x \in c_0 : \forall i \in \mathbb{N} : |x_i| \leq \frac{1}{2}(1 + \frac{1}{i}) \text{ and } |x_n| = -\frac{1}{2}(1 + \frac{1}{n})\}.$$



One has  $a_n^+ \cap a_n^- = \emptyset$  for all  $n \in \mathbb{N}$ . But for every  $z \in \{-, +\}^{\mathbb{N}}$  the choice  $a_n^{z(n)} \in \mathcal{C}_n$  has  $x \in \bigcap_n a_n^{z(n)} \neq \emptyset$  for  $x_i = z(i)\frac{1}{2i}$ .

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